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INSEPARABILITY GRAPHS OF ORIENTED MATROIDS

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We determine the inseparability graphs of uniform oriented matroids and of graphic oriented matroids. For any r, n such that $4 \leqq r \leqq n-3$, examples of rank r uniform oriented matroids on n elements with a given inseparability graph are obtained by simple constructions of polytopes having prescribed separation properties.

1. Introduction

A combinatorial setting for the study of properties of dependence in vector spaces over an ordered field, or of directed graphs, is provided by the theory of oriented matroids. The fundamental concepts and the basic results concerning this theory have been introduced by R. G. Bland and M. Las Vergnas [2], [8] and J. Folkman and J. Lawrence [4]. Many properties, known for vector spaces or for directed graphs have been extended to oriented matroids. On the other hand, and it is one aim of this paper, the oriented matroid theory reveals useful for constructing sets of points or convex polytopes having prescribed combinatorial properties (Theorem 3.6, see also [10]).

Let M denote a rank r oriented matroid on E, $|E|=n$. Two distinct elements x and y of E are said to be *inseparable* in M if x and y have always the same sign or always an opposite sign in all circuits of M containing them. The graph IG (M) on E which edges are the inseparable pairs of M is called the *inseparability graph* of M. (See [2].)

Clearly IG (M) is invariant under sign reversals of M, hence the study of the inseparability graphs appears as a natural tool for the determination of the orientation class of a matroid. Actually we have proved in [11] that if M is uniform, then M is determined up to sign reversals by the list of inseparability graphs IG $(M(A))$, $A \subseteqq E$.

The main purpose of this paper is to determine the inseparability graphs for certain classes of oriented matroids.

In Section 3, we describe all possible inseparability graphs of rank r uniform oriented matroids on n elements. We first give two conditions satisfied by any of these graphs (Theorem 3.2 [3] and Theorem 3.3).

We then show that these two conditions are sufficient by constructing representable rank r uniform oriented matroids having a given admissible graph G as inseparability graph (Theorem 3.6). More precisely, using the concatenation

AMS subject classification (1980): 05 B 35 operation [10], we construct for every r, n with $4 \leqq r \leqq n-3$ a simplicial polytope of $\mathbf{R}^{r-1}$ with vertex set V, $|V|=n$, such that $\mathrm{IG}(\mathrm{Aff}(V))=G$.

The notion of separability in a representable rank r oriented matroid $\mathrm{Aff}(V)$ is of course related to the usual notion of separability in $\mathbf{R}^{r-1}$. Any two elements x and y of V are inseparable in $\mathrm{Aff}(V)$ if no hyperplane spanned by a subset of $V \setminus \{x, y\}$ (strictly) separates x and y, or on the contrary if every such hyperplane separates x and y.

Section 4 deals with graphic oriented matroids. We relate the problem of finding inseparable pairs in graphic oriented matroids to the problem of finding disjoint paths between given pairs of points in a graph (Observation 4.1). We then show that the inseparability graph of the oriented cycle matroid of a graph H is the line-graph $L(H)$ of H if H is 4-connected and non-planar (Theorem 4.3), and that the inseparability graph is $L(H) \cup L(H^*)$ if H is 3-connected and planar, where H^* denotes the topological dual of H (Theorem 4.6).

2. Definitions and notation

Our notation concerning oriented matroids follows closely the notation of [2].

A *signed set* is a set X together with a partition into two distinguished subsets X^+ and X^-. The *opposite* of X is the signed set $-X$ with $(-X)^+=X^-$ and $(-X)^-=X^+$.

An *oriented matroid* M on a finite set E is defined by its collection $\mathscr{C}$ of *signed circuits*, i.e. signed subsets of E satisfying the following two properties:

2.1. For all $C \in \mathscr{C}$, $C \neq \emptyset$ and $-C \in \mathscr{C}$ and for all $C_1, C_2 \in \mathscr{C}$, $C_2 \subseteqq C_1$ implies $C_2=C_1$ or $-C_1$.

2.2. *(Elimination property.)* For all $C_1, C_2 \in \mathscr{C}$ such that $C_1 \neq -C_2$ and for all $x \in C_1^+ \cap C_2^-$, there exists $C_3 \in \mathscr{C}$ such that $C_3^+ \subseteqq (C_1^+ \cup C_2^+) \setminus x$ and $C_3^- \subseteqq (C_1^- \cup C_2^-) \setminus x$.

The *underlying matroid* $\underline{M}$ of M is obtained from M by forgetting signs. The *rank* of M is the rank of $\underline{M}$. The cocircuits of M can be signed in a natural way in order to obtain an oriented matroid M^* having the dual $\underline{M}^*$ as underlying matroid.

Let $A \subseteqq E$. The *deletion* of M by A is the oriented matroid $M \setminus A$ on $E \setminus A$ with circuit set $\{C \in \mathscr{C}, C \cap A = \emptyset\}$. The *restriction* of M to A is the oriented matroid $M(A)=M \setminus (E \setminus A)$.

We denote by ${}_{\bar{A}}M$ the oriented matroid obtained from M by *reversing signs* on A, i.e. with circuit set: $\{C'={}_{\bar{A}}C,\ C \in \mathscr{C}\}$, where $({}_{\bar{A}}C)^+=(C^+ \setminus A) \cup (C^- \cap A)$ and $({}_{\bar{A}}C)^-=(C^- \setminus A) \cup (C^+ \cap A)$.

The equivalence classes for the relation: $M \sim M'$ if and only if there exists $A \subseteqq E$ such that $M'={}_{\bar{A}}M$, are called the *orientation classes*.

As defined in our introduction, two distinct elements x and y of E are *inseparable* in M if x and y have always the same sign or always an opposite sign in all circuits of M containing them. The elements x and y are *separable* otherwise.

The *inseparability graph* of M is the graph $\mathrm{IG}(M)$ on E with edges the inseparable pairs.

The following two properties are simple but essential in what follows:

Property 2.3. *For every* $A \subseteq E$, $\mathrm{IG}(_{\bar{A}}M) = \mathrm{IG}(M)$.

Property 2.4. $\mathrm{IG}(M^*) = \mathrm{IG}(M)$.

We recall a fundamental example of oriented matroid [2, Example 3.5]. Let V denote a finite set of points spanning $\mathbf{R}^d$ and let $\mathscr{C}$ be the collection of signed subsets C of V which are inclusion-minimal with the property:

2.5. $C \neq \emptyset$ and there is an affine combination $\sum_{x \in C} \beta(x) \cdot x = 0$,

$$\sum_{x \in C} \beta(x) = 0 \quad \text{such that} \quad C^+ = \{x \in C, \beta(x) > 0\}$$

and

$$C^- = \{x \in C, \beta(x) < 0\}.$$

Then $\mathscr{C}$ is the circuit set of an oriented matroid of rank $d+1$ on V, called the *oriented matroid of affine dependencies* of V and denoted by $\mathrm{Aff}(V)$. In particular, $\mathrm{Aff}(V)$ is *acyclic* which means that $\mathrm{Aff}(V)$ has no circuit C with $C^- = \emptyset$.

A *representable oriented matroid* [2] can be defined as any oriented matroid isomorphic to $_{\bar{A}}\mathrm{Aff}(V)$ for some $A \subseteq V \subseteq \mathbf{R}^d$.

We will also need a construction introduced by J. Lawrence and L. Weinberg [10]. We suppose in what follows that E is the set $\{e_1, e_2, \dots, e_n\}$ totally ordered by $e_1 < e_2 < \dots < e_n$.

Let M_1 and M_2 be two uniform oriented matroids on E of respective ranks r_1 and r_2, with $r_1 + r_2 \leqq n$. Suppose that C_1 and C_2 are circuits of M_1 and M_2 respectively, satisfying:

2.6. There is an index i such that $C_1 \subseteq \{e_1, e_2, \dots, e_i\}$ and $C_2 \subseteq \{e_i, e_{i+1}, \dots, e_n\}$, and either $e_i \in C_1^+ \cap C_2^+$ or $e_i \in C_1^- \cap C_2^-$.

Let C be the signed set defined by $C^+ = C_1^+ \cup C_2^+$ and $C^- = C_1^- \cup C_2^-$. The set of signed sets obtained in this way is the circuit set of a uniform oriented matroid M of rank $r_1 + r_2$ on E called the *concatenation* of M_1 and M_2. The concatenation is a particular construction of *union* of oriented matroids, a notion introduced in [9] and developped in [10]. Therefore, we will denote M by $M_1 \vee M_2$. Of course, this notation depends on the order chosen on E.

We further remark that $M_1 \vee M_2$ is completely determined by the knowledge of $M_1(\{e_1, e_2, \dots, e_{n-r_2}\})$ and $M_2(\{e_{r_1+1}, e_{r_1+2}, \dots, e_n\})$.

Thus, if M_1 and M_2 now denote uniform oriented matroids of rank r_1 on $\{e_1, e_2, \dots, e_{n-r_2}\}$ and of rank r_2 on $\{e_{r_1+1}, e_{r_1+2}, \dots, e_n\}$ respectively, we can define $M_1 \vee M_2$ as the oriented matroid $\tilde{M}_1 \vee \tilde{M}_2$, where $\tilde{M}_1$ and $\tilde{M}_2$ denote any uniform oriented matroids on E, of respective ranks r_1 and r_2, such that:

$$\tilde{M}_1(\{e_1, e_2, \dots, e_{n-r_2}\}) = M_1 \quad \text{and} \quad \tilde{M}_2(\{e_{r_1+1}, e_{r_1+2}, \dots, e_n\}) = M_2.$$

Finally we will sometimes use the following notation: $e_1\overline{e_2}\overline{e_3}e_4\overline{e_5}$ denotes the signed set X with $X^+ = \{e_1, e_4\}$ and $X^- = \{e_2, e_3, e_5\}$.

Basic notions concerning graphs and polytopes are also used in this paper, for which the reader is referred to [1] and [6] respectively.

We precise however some notation. A chain (resp. a cycle) with n vertices will be denoted by P_n (resp. C_n). For any two graphs G and G', $G+G'$ denotes the disjoint union of G and G'. For any two graphs G and G' with the same vertex set V, $G \cup G'$ denotes the graph on V, any edge of which is either an edge of G or an edge of G'. Finally, if there is no ambiguity on the number n of vertices, we denote by $\emptyset$ the empty graph on n vertices, i.e. the graph with no edges on n vertices.

3. Inseparability graphs of uniform oriented matroids

Our goal in this section is to determine all possible inseparability graphs for rank r uniform oriented matroids on n elements, n and r being fixed.

We first recall two results of R. Cordovil and P. Duchet:

Theorem 3.1 [3]. *Let M be a rank r uniform oriented matroid on E, $|E|=n$.*
If $r=1$, then $\mathrm{IG}(M)$ is the complete graph.
If $r=2$, then $\mathrm{IG}(M)$ is a n-cycle.

Theorem 3.2 [3]. *Let M be a rank r uniform oriented matroid on E, $|E|=n$ with $2 \leqq r \leqq n-2$.*

Then $\mathrm{IG}(M)$ is either a n-cycle, or a disjoint union of k chains, $k \geqq 2$.

We now give another necessary condition for inseparability graphs of uniform oriented matroids:

Theorem 3.3. *Let M be a rank r uniform oriented matroid on E, $|E|=n$, with $2 \leqq r \leqq n-2$.*

Let $\{x, y, z, t\} \subseteqq E$ and suppose that the subgraph of $\mathrm{IG}(M)$ induced by $E \setminus \{x, y, z, t\}$ is a $(n-4)$-chain.

Then the subgraph of $\mathrm{IG}(M)$ induced by $\{x, y, z, t\}$ has at least two edges.

Proof. We proceed by induction on the couples (r, n) lexicographically ordered, i.e. $(r', n') \prec (r, n)$ if and only if $r' < r$ or $r' = r$ and $n' < n$.

If $r=2$, $\mathrm{IG}(M)$ is a n-cycle by Theorem 2.1 and the result is immediate.

The oriented matroids having the parameters $(r, n)=(3, 6)$ are all representable [5]. It is easily checked that there exist exactly 4 orientation classes for $(r, n)=(3, 6)$ whose corresponding inseparability graphs are C_6, P_3+P_3, $P_2+P_2+P_2$ and $\emptyset$ respectively (see Figure 1). Thus, the result is also true for $(r, n)=(3, 6)$.

If $n<2r$, then $(n-r, n) \prec (r, n)$. We may apply the induction hypothesis to M^* and conclude by Property 2.4. We assume in what follows that $n \geqq 2r \leqq 6$ and that $(r, n) \neq (3, 6)$. We denote by $e_1, e_2, \ldots, e_{n-4}$ the $(n-4)$-chain $\mathrm{IG}(M) \setminus \{x, y, z, t\}$.

Lemma 3.4. *Let $1 \leqq i < j \leqq n-4$. Then x and y are inseparable in $M \setminus e_i$ if and only if x and y are inseparable in $M \setminus e_j$.*

We only need to prove Lemma 3.4 for consecutive indices i and j. Without loss of generality, we may assume by Property 2.3 that x and y have the same sign in all circuits of $M \setminus e_i$ containing them.

Suppose that x and y are separable in $M \setminus e_j$. There exists a circuit C in $M \setminus e_j$ such that $x \in C^+$ and $y \in C^-$. Clearly, $e_i \in C$. Let C' be the circuit of M such that $\{x, y\} \subseteqq C'^+$ and (as a set) $C' = C \cup e_j \setminus e_i$. By eliminating y between C and C' and x between C and $-C'$, we obtain that e_i and e_j are separable in M: a contradiction.

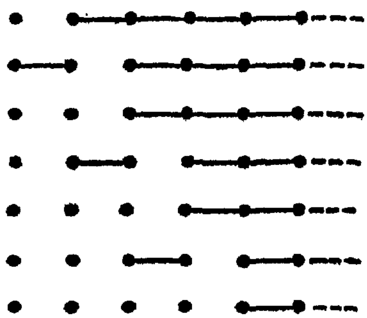
We now apply the induction hypothesis to $M\setminus e_1$. There are at least two edges in the subgraph of IG $(M\setminus e_1)$ induced by $\{x, y, z, t\}$. Suppose for instance that $\{x, y\}$ is one of them. Up to a sign reversal we may assume that x and y have the same sign in all circuits of $M\setminus e_i$ containing them, for every i, $1\leqq i\leqq n-4$.

If x and y are separable in M, x and y have an opposite sign in some circuit C of M. Thus, C contains each e_i and $|C|\geqq n-2\geqq 2r-2$.

As $|C|=r+1$, we get $r\leqq 2$ or $r=3$ and $n=6$: a contradiction. ∎

By applying Theorem 3.3 to suitably chosen $(n-4)$-chains, we obtain:

Corollary 3.5. *Let M be a rank r uniform oriented matroid on E, $|E|=n$. Then* IG (M) *is none of the following graphs:*



P_1+P_{n-1}

P_2+P_{n-2}

$P_1+P_1+P_{n-2}$

$P_1+P_2+P_{n-3}$

$P_1+P_1+P_1+P_{n-3}$

$P_1+P_1+P_2+P_{n-4}$

$P_1+P_1+P_1+P_1+P_{n-4}$.

We call *admissible graph* any graph which is either a cycle, or which is the disjoint union of $k\geqq 2$ chains and is none of the 7 forbidden graphs of Corollary 3.5.

The following theorem shows that the necessary conditions given in Theorem 3.2 and Corollary 3.5 are in fact sufficient.

Theorem 3.6. *Let r, n be integers satisfying $3\leqq r\leqq n-3$ and let G be an admissible graph with n vertices.*

Then, there exists a representable rank r uniform oriented matroid M such that IG $(M)=G$.

Moreover, if $r\geqq 4$, we can take $M=\mathrm{Aff}\,(V)$, where V is the vertex set of a simplicial polytope of $\mathbf{R}^{r-1}$.

Proof. 3.6.1. Case $r=3$.

For each admissible graph G on n vertices, we construct a set V of n points in general position in $\mathbf{R}^2$ such that IG $(\mathrm{Aff}\,(V))=G$.

On Figure 1, we only give examples for minimal admissible graphs, i.e. $G=\emptyset$ or $G\neq\emptyset$ and $G\setminus e$ is non-admissible for each edge e of G. The other admissible graphs are obtained by adding new points (in general position) on the dotted curves drawn on the figures.

3.6.2. Case $G=\emptyset$.

We denote by $A(e_1, e_2, \ldots, e_p)$ the *alternating* rank 2 uniform oriented matroid on p elements, i.e. every circuit $C=\{e_i, e_j, e_k\}$, $i<j<k$, has the signature $e_i\bar{e}_j e_k$ or $\bar{e}_i e_j \bar{e}_k$ [2, Example 3.8].

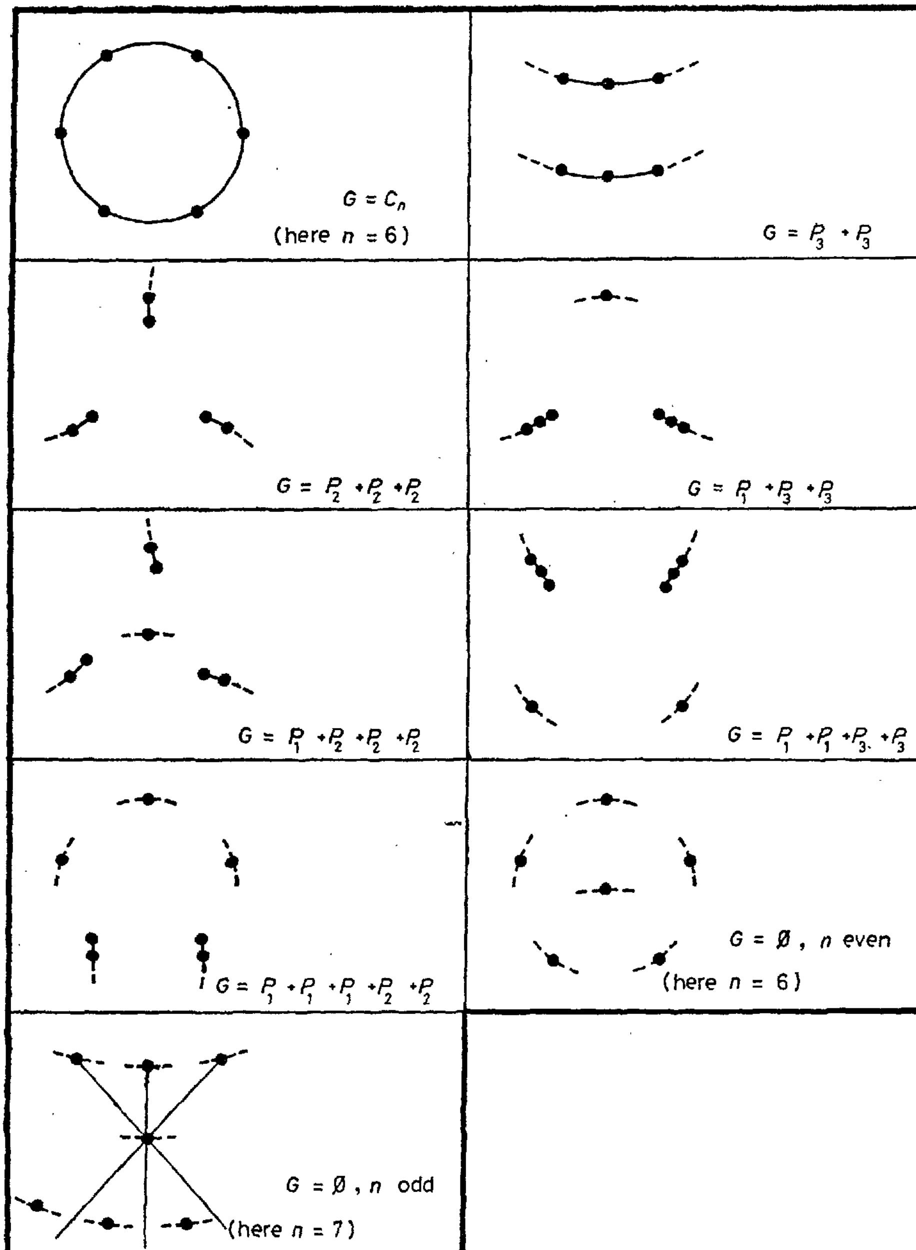

$G = C_n$
(here $n = 6$)
$G = P_3 + P_3$
$G = P_2 + P_2 + P_2$
$G = P_1 + P_3 + P_3$
$G = P_1 + P_2 + P_2 + P_2$
$G = P_1 + P_1 + P_3 + P_3$
$G = P_1 + P_1 + P_1 + P_2 + P_2$
$G = \emptyset$, n even
(here $n = 6$)
$G = \emptyset$, n odd
(here $n = 7$)

Fig. 1

Let $E=\{e_1, e_2, ..., e_n\}$ with $n\geqq 7$, ordered by $e_1<e_2<...<e_n$. We define two rank 2 uniform oriented matroids M_1 and M_2, on $\{e_1, e_2, ..., e_{n-2}\}$ and on $\{e_3, e_4, ..., e_n\}$ respectively, by:

$$M_1 = A\left(e_1, e_{n-2}, e_2, e_{n-3}, e_3, ..., e_{\left[\frac{n}{2}\right]}\right)$$

and

$$M_2 = A\left(e_n, e_3, e_{n-1}, e_4, e_{n-2}, ..., e_{n-\left[\frac{n}{2}\right]+1}\right).$$

3.6.2.1. $G=\emptyset$, $r=4$.

Let $M=M_1\vee M_2$: M is a rank 4 uniform oriented matroid on E, which is representable because both M_1 and M_2 are representable.

Moreover, as M_1 and M_2 are acyclic, every circuit of M clearly satisfies $|C^+|\geqq 2$ and $|C^-|\geqq 2$, hence M is acyclic and has no interior point. Thus, $M\cong \mathrm{Aff}(V)$, where V denotes the vertex set of a polytope of $\mathbf{R}^3$, which is simplicial because M is uniform. We show that $\mathrm{IG}(M)=\emptyset$. Let $1\leqq i<j\leqq n$.

If $j\leqq n-2$, e_i and e_j are inseparable in M_1 if and only if $(i,j)=\left(1, \left[\frac{n}{2}\right]\right)$ or if $i+j=n-1$ or n. If $i\geqq 3$, e_i and e_j are inseparable in M_2 if and only if $(i,j)=\left(n-\left[\frac{n}{2}\right]+1, n\right)$ or if $i+j=n+2$ or $n+3$. In particular, if $3\leqq i<j\leqq n-2$, e_i and e_j are separable in M_1 or M_2, hence also separable in M. In the remaining cases, we may assume by symmetry that $i=1$ or 2.

If $j\leqq 3$, e_i and e_j are separable in M_1, hence also in M.

If $4\leqq j\leqq n-2$, e_3 and e_j are separable in M_2: there exist circuits C, C' of M_2 such that $\{e_3, e_j\}\subseteqq C^+$, $e_3\in C'^+$ and $e_j\in C'^-$. The concatenations of $e_1\overline{e_2}e_3$ and C, and of $e_1\overline{e_2}e_3$ and C' show that e_i and e_j are separable in M.

If $j\geqq n-1$, we conclude by looking at the following circuits of M:

$$e_1\overline{e_2}e_3\overline{e_{n-1}e_n},\quad e_1\overline{e_2}e_{n-3}\overline{e_{n-2}}e_{n-1}\quad\text{and}\quad e_1\overline{e_2}e_{n-3}\overline{e_{n-2}}e_n.$$

3.6.2.2. Case $r\geqq 5$.

Using an inductive argument, there is a representable acyclic uniform oriented matroid M' of rank $r-2$ on $\{e_1, e_2, ..., e_{n-2}\}$ with $\mathrm{IG}(M')=\emptyset$.

The same method as in 3.6.2.1 shows that $M=M'\vee M_2$ satisfies $\mathrm{IG}(M)=\emptyset$ and $M\cong\mathrm{Aff}(V)$ where V is the vertex set of a simplicial polytope of $\mathbf{R}^{r-1}$. We notice that Theorem C of [3] is a straightforward consequence of Case 3.6.2.

3.6.3. Case $(r,n)=(4,8)$ and $G=P_1+P_1+P_3+P_3$.

The following uniform oriented matroid is easily seen to fit:

$$A(e_1, e_2, e_3, e_6, e_4, e_5)\vee A(e_4, e_3, e_5, e_6, e_7, e_8).$$

3.6.4. Case $G=C_n$.

We may take for M any alternating rank r uniform oriented matroid on E. We have $M\cong\mathrm{Aff}(V)$, where V is the vertex set of a cyclic polytope of $\mathbf{R}^{r-1}$ [3].

3.6.5. General case.

We suppose $4\leqq r\leqq n-3$ and we proceed by induction on r, assuming $G\neq\emptyset$, $G\neq C_n$ and $G\neq P_1+P_1+P_3+P_3$. Under these conditions, we remark that G possesses at least one pendant vertex, say e_n, such that $G\setminus e_n$ is also an admissible graph. We index the remaining vertices $e_1, e_2, \ldots, e_{n-1}$ in such a way that if $\{e_i, e_j\}$ with $1 \leq i < j \leq n$, is an edge in G, then $j = i+1$.

By the induction hypothesis, there is a representable acyclic uniform oriented matroid M' of rank $r-1$ on $E \setminus e_n$ such that $\mathrm{IG}(M') = G \setminus e_n$.

We include to the induction hypothesis two technical conditions, which are easily seen to be satisfied when $r-1=3$ (see Figure 1.)

We suppose that if e_i and e_j are inseparable in M', then they have an opposite sign in all circuits of M' containing them. We also assume that for every circuit $C = \{e_{i_1}, e_{i_2}, \ldots, e_{i_r}\}$ of M', with $i_1 < i_2 < \ldots < i_r$, we have $C^- \neq e_{i_r}$. (In other words, e_i is an extreme point of $M'(\{e_1, e_2, \ldots, e_i\})$ for every i, $r \leq i \leq n-1$.)

Let N be the unique acyclic uniform oriented matroid of rank 1 on $\{e_2, e_3, \ldots, e_n\}$ (the circuits of N are $e_i\bar{e}_j$, $i \neq j$).

Let $M = M' \vee N$. M' and N being uniform, acyclic and representable, M is also uniform (of rank r), acyclic and representable. Moreover, the above conditions on M' imply $|C^+| \geq 2$ for every circuit C of M, hence $M \cong \mathrm{Aff}(V)$, where V is the vertex set of a simplicial polytope of $\mathbf{R}^{r-1}$. We show that $\mathrm{IG}(M) = G$. Let $1 \leq i < j \leq n$. If $\{e_i, e_j\}$ is an edge of G, then $j = i+1$.

As e_i and e_{i+1} have an opposite sign in all circuits of M' or N containing them, they also have an opposite sign in all circuits of M containing them, hence G is a subgraph of $\mathrm{IG}(M)$.

If $\{e_i, e_j\}$ is not an edge of G, and $j \leq n-1$, e_i and e_j are separable in M', hence in M.

Finally, if $\{e_i, e_n\}$ is not an edge in G, then $i \leq n-2$. If $\{e_i, e_{n-1}\}$ is not an edge in G, then e_i and e_{n-1} are separable in M' and as in 2.6.2.1, e_i and e_n are separable in M, by concatenation with $e_{n-1}\bar{e}_n$. If $\{e_i, e_{n-1}\}$ is an edge of G, then e_i and e_n are separable in M otherwise $\mathrm{IG}(M)$ would contain the 3-cycle e_i, e_{n-1}, e_n, e_i. ∎

Remark 3.7. When $M = \mathrm{Aff}(V)$, two elements e_i and e_j of V have an opposite sign in every circuit of M, if and only if there is no hyperplane spanned by a subset of $V \setminus \{e_i, e_j\}$ that strictly separates e_i and e_j.

The above proof (particularly Case 3.6.5) shows that Theorem 3.5 remains valid for points in general position in $\mathbf{R}^d$, $d \geq 3$, if the notion of separability for oriented matroids is replaced by the usual notion of separability in $\mathbf{R}^d$.

4. Inseparability graphs of graphic oriented matroids

In all this section, H denotes a 2-connected graph with vertex set V and edge set E. The line-graph of H is denoted by $L(H)$.

It is proved in [2] that the cycle matroid $\mathbf{C}(H)$ of H has exactly one orientation class, each orientation of $\mathbf{C}(H)$ corresponding to some orientation of the edges of E. Thus, the inseparability graph of (any orientation of) $\mathbf{C}(H)$ only depends on the structure of H and will be denoted by $\mathrm{IG}(H)$. We also say that two distinct edges of E are inseparable in $\mathbf{C}(H)$ if they are inseparable in any orientation of $\mathbf{C}(H)$.

Let $\{x, y, x', y'\} \subseteq V$. Using the notation of [13], we say that H has a (x, y, x', y')-*linkage* if there exist two vertex-disjoint chains P and P' in H, P connecting x with x' and P' connecting y with y'.

Observation 4.1. *Let* $e = \{x, y\}$, $e' = \{x', y'\}$ *be two distinct edges of* H. *Then,* e *and*

e' are inseparable in $\mathbf{C}(H)$ *if and only if H has a* (x, y, x', y')*-linkage and a* (x, y, y', x')*-linkage.*

In particular, x, y, x' and y' have to be pairwise distinct and we get:

Proposition 4.2. *If H is 2-connected, then* $\mathrm{IG}(H)$ *contains* $L(H)$.

With stronger conditions on H, a converse to this proposition can be stated:

Theorem 4.3. *If H is a 4-connected non-planar graph, then* $\mathrm{IG}(H)=L(H)$.

We observe that Theorem 4.3 is a straightforward consequence of the following result of Jung:

Theorem 4.4 [7]. *If H is 4-connected and if* $\{x, y, x', y'\}\subseteq V$, *then H has a* (x, y, x', y')*-linkage, unless H is planar and possesses a facial cycle containing* x, y, x', y' *in that cyclic order.*

Remark 4.5. Theorem 4.3 may be false if we only assume that H is 3-connected. For instance e and e' are inseparable in $\mathbf{C}(H)$ whenever there is an edge e'' of E such that $\{e, e', e''\}$ is a cocycle of H.

In the general case, good characterizations for the existence of a (x, y, x', y')-linkage in a graph, have been found independently by Seymour [12] and Thomassen [13].

These results can of course be used to determine $\mathrm{IG}(H)$ but unfortunately rather technical statements are obtained. For 3-connected planar graphs however, $\mathrm{IG}(H)$ has a very simple expression, for which we give a short proof.

Let H be a 3-connected planar graph and let H^* be the topological dual of H (H^* is unique up to isomorphism). There is a natural 1—1 correspondence between the edges in H and those in H^* and we denote by the same letter an edge in H and the corresponding edge in H^*.

Theorem 4.6. *If H is a 3-connected planar graph, then:* $\mathrm{IG}(H)=L(H)\cup L(H^*)$. *Equivalently, two distinct edges e and e′ are inseparable in* $\mathbf{C}(H)$ *if and only if e and e′ have a vertex in common, or if a facial cycle of H contains both e and e′.*

Proof. We have already seen that $\mathrm{IG}(H)$ contains $L(H)$. On the other hand, as $\mathbf{C}(H)$ and $(\mathbf{C}(H))^*$ have the same inseparability graph by Property 2.4 and since $(\mathbf{C}(H))^*=\mathbf{C}(H^*)$, $\mathrm{IG}(H)$ also contains $L(H^*)$.

Conversely, let e and e' be two edges of H having no vertex of V in common and such that no facial cycle of H contains both e and e'.

Using Steinitz's theorem [6], we may consider H as the 1-skeleton of a polytope $\mathscr{P}$ of $\mathbf{R}^3$. Thus, we may identify e and e' with edges of $\mathscr{P}$, say $[x, x']$ and $[y, y']$ respectively. We remark that there always exist a plane P of $\mathbf{R}^3$ such that $\{x, x'\}\subsetneq P^+$ and $\{y, y'\}\subsetneq P^+$ where P^+ and P^- denote the closed half-planes defined by P.

The above conditions on e and e' imply that $\mathscr{P}\cap P^+$ and $\mathscr{P}\cap P^-$ are two polytopes of $\mathbf{R}^3$, $\mathscr{P}\cap P$ being a face of both.

Moreover, x and x' are two vertices of $\mathscr{P}\cap P^+$. By Menger's theorem [1], there exist three chains of $\mathscr{P}\cap P^+$ connecting x with x', any two of which having only x and x' in common. One of them, say C, does not meet the face $\mathscr{P}\cap P$, except perhaps in x or x', hence this chain in $\mathscr{P}\cap P^+$ is also a chain in $\mathscr{P}$. We get similarly a chain C' connecting y and y' in $\mathscr{P}\cap P^-$, which is also a chain in $\mathscr{P}$. As C

and C' are vertex-disjoint, we have constructed a (x, y, x', y')-linkage in H. By exchanging the roles of x' and y', H also possesses a (x, y, y', x')-linkage and we conclude by Observation 4.1. ∎

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